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Duality Relations for Asymmetric Exclusion Processes

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We derive duality relations for a class of $U_q[SU(2)]$-symmetric stochastic processes, including among others the asymmetric exclusion process in one dimension. Like the known duality relations for symmetric hopping processes, these relations express certain m-point correlation functions in N-particle systems ($N \geqslant m$) in terms of sums of correlation functions of the same system but with only m particles. For the totally asymmetric case we obtain exact expressions for some boundary density correlation functions. The dynamical exponent for these correlators is $z = 2$, which is different from the dynamical exponent for bulk density correlations, which is known to be $z = 3/2$.

KEY WORDS: Asymmetric exclusion processes; duality relations; quantum algebra; correlation functions; dynamical scaling.

1. INTRODUCTION

Duality relations play an important role in the study of stochastic models of interacting particles.[1] The basic idea behind duality in this context is to relate a given stochastic process to another, dual one, in such a way that quantities which one may compute for one process are expressed in terms of quantities obtained from the dual process. In many cases it happens that the dual process is more tractable than the original one and calculations simplify considerably. An important example is the symmetric exclusion process describing lattice diffusion of identical particles with hard-core

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repulsion. Here duality relates m-point equal-time density correlation functions of a system with $N \geqslant m$ particles (for arbitrary initial conditions) to sums of correlation functions of a system with only m particles. So, for instance, the time evolution of the average particle density at some lattice site k of an N-particle system is completely given by the dynamics of a one-particle system and can therefore be calculated exactly.[2]

Recently there has been much interest in stochastic processes which can be mapped to quantum lattice models or to vertex models. In this mapping the set of allowed configurations of the system becomes a vector space (where each configuration is a basis vector) and the time evolution of the (stochastic) variables is defined in terms of a "Hamilton" operator or transfer matrix acting on this space. A particularly interesting subclass of such systems are those where the dynamics is mapped to integrable quantum chains in one dimension,[3,4] to integrable two-dimensional vertex models,[10-14] or to $SU(N)$-symmetric quantum Hamiltonians in any dimension,[15,16] an example of which is the symmetric exclusion process mentioned above. Indeed, in the latter case it is possible to rederive the duality relations from the $SU(2)$ symmetry of the problem alone and thereby obtain new duality relations for other $SU(2)$-symmetric systems such as diffusion with partial exclusion as well as relations involving correlation functions for different times.[15] Examples of integrable models which can be studied using the quantum Hamiltonian formalism are the asymmetric exclusion process and other driven lattice gases, which in one dimension are related to growth models.[17]

In this paper we study various versions of asymmetric exclusion processes describing driven diffusive lattice gases with space-dependent hopping rates in one dimension (see below). In its simplest version this model is equivalent to an RSOS-type model of a growing interface.[18,19] The continuum limit of this model is believed to be in the universality class of the KPZ equation.[20] The model also gives rise to a discrete version of the noisy Burgers equation and is therefore of relevance for the study of shock fronts in one-dimensional media.[11,21-25] In yet another mapping the totally asymmetric model may be used to describe directed polymers in disordered media.[17,26-29] More involved versions of asymmetric exclusion processes have been used as models for freeway traffic.[30-32]

Unfortunately there are no simple duality relations for density correlation functions for this model as one has for the undriven case, i.e., symmetric exclusion, and very few *exact* results on the dynamics of the driven lattice gas are known. One of these is a result by Gwa and Spohn,[4] who proved that in a half-filled system with periodic boundary conditions and hopping in only one direction the lowest energy gap of the time evolution operator vanishes with the size L of the system as $L^{-3/2}$. This was later

shown to be true for the lowest energy gaps at any finite particle density.[9] Both results are consistent with the expected dynamical exponent $z = 3/2$ of the model.

Here we study the system with reflective boundary conditions, i.e., particles cannot enter or leave the system. We show that if the asymmetry in left- and right-hopping probabilities is spatially constant, the system is symmetric under the action of the quantum deformation $U_q[SU(2)]$ of $SU(2)$,[33-35] where the deformation parameter q parametrizes the hopping asymmetry. We will use this symmetry to introduce the lattice analogue of the Hopf–Cole transformation[36] on the KPZ equation. This transformation gives rise to duality relations generalizing those known for the symmetric model. These relations for the asymmetric case do not involve density correlators, but exponentials of integrated densities (see below). In the continuum limit this quantity may be interpreted as the partition function of a polymer chain in two dimensions.[17] It also plays an important role in the study of interfaces in disordered system.[37,38] The simplest of the duality relations was derived previously in ref. 7. Here we derive the full set of relations and we show how to obtain similar relations for correlation functions for different times.

In the totally asymmetric limiting case where particles hop only in one direction (say to the right) these operators become products of density operators acting on neighboring sites. Thus one obtains exact expressions for the probability of finding p neighboring sites occupied where the fist of these p sites is the leftmost site initially occupied.

The paper is organized as follows. In Section 2 we set up the formalism and define the model. We briefly review some of its symmetries, in particular its symmetry under the quantum algebra $U_q[SU(2)]$. In Section 3 we derive the duality relations for the partially asymmetric case ($q \neq 0, 1, \infty$). In Section 4 we treat separately the totally asymmetric model where particles hop only to the right, corresponding to $q = 0$. In Section 5 we describe other $U_q[SU(2)]$-symmetric driven systems where the same relations hold. In Section 6 we give a brief summary of our results and point out some open problems.

2. THE $U_q[SU(2)]$-INVARIANT ASYMMETRIC EXCLUSION PROCESS

Let us first define the simplest version of the models we are going to study. We consider a one-dimensional system where each lattice site can be occupied by at most one particle. The state of the system at some time t is given by (stochastic) occupation numbers $\mathbf{n} = \{n_k\}$, where $n_k = 0, 1$ and k is the number of a lattice site and runs from 1 to L. The probability of

finding a given configuration $\mathbf{n}$ at time t is given by the probability distribution $F(\mathbf{n}, t)$ which satisfies $\sum_{\mathbf{n}} F(\mathbf{n}, t) = 1$ (conservation of probability).

The stochastic dynamics of the system is given by a master equation for the probability distribution $F(\mathbf{n}, t)$ which can be mapped onto a Schrödinger equation[39-41] with a quantum Hamiltonian H:

$$\partial_t |F(t)\rangle = -H |F(t)\rangle \tag{2.1}$$

Here $|F(t)\rangle = \sum_{\mathbf{n}} F(\mathbf{n}, t) |\mathbf{n}\rangle$, where $|\mathbf{n}\rangle$ is a basis vector representing the configuration $\mathbf{n}$. The Hamiltonian H acts on the 2^L-dimensional vector space spanned by these basis vectors, which together with their transposed vectors $\langle \mathbf{n}|$ form an orthonormal basis. Thus $F(\mathbf{n}, t) = \langle \mathbf{n} | F(t) \rangle$, and by explicitly defining the Hamiltonian H the usual master equation can be recovered from (2.1). Conservation of probability translates into $\langle s | F(t) \rangle = 1$, where $\langle s| = \sum_{\mathbf{n}} \langle \mathbf{n}|$ is the sum over all left basis vectors. This implies the requirement $\langle s| H = 0$ on H.

We shall assume that particles hop with space-dependent probability rates α_k from site $k+1$ to site k and with rates β_k from site k to $k+1$. The Hamiltonian is then given by

$$H = -\sum_{k=1}^{L-1} [\alpha_k (s_k^- s_{k+1}^+ - (1-n_k) n_{k+1}) + \beta_k (s_k^+ s_{k+1}^- - n_k (1-n_{k+1}))] \tag{2.2}$$

$$\equiv \sum_{k=1}^{L-1} \mu_k u_{k,k+1} \tag{2.3}$$

where

$$u_{k,k+1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q_k & -q_k^{-1} & 0 \\ 0 & -q_k & q_k^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{k,k+1} \tag{2.4}$$

acts nontrivially only on nearest neighbor sites k and $k+1$ and

$$q_k = \sqrt{\frac{\alpha_k}{\beta_k}}, \qquad \mu_k = \sqrt{\alpha_k \beta_k} \tag{2.5}$$

In our convention $s_k^- = (\sigma_k^x - i\sigma_k^y)/2$ creates a particle at site k, while $s_k^+ = (\sigma_k^x + i\sigma_k^y)/2$ annihilates a particle at site k ($\sigma^{x,y}$ are the usual Pauli

matrices). $n_k = (1 - \sigma_k^z)/2$ is the number operator. In the basis chosen here one has

$$s^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad s^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.6}$$

and expressions of the form A_k for some matrix A are understood as tensor products $1 \otimes \cdots \otimes 1 \otimes A \otimes 1 \otimes \cdots \otimes 1$ with A at the kth position in the product and 1 being the two-by-two unit matrix.

The logarithm of the ratio q_k^2 of the hopping rates represents the asymmetry of the model and may be thought of as resulting from some driving force. The product $\mu_k = \sqrt{\alpha_k \beta_k}$ gives the (relative) probability that a hopping process occurs between sites k and $k+1$ and may therefore be considered as a measure for the (space-dependent) mobility of the particles. We assume the driving foce to be independent of the position, i.e., $q_k \equiv q$, $\forall k$. Here $q = 1$ corresponds to diffusion without driving force (symmetric diffusion), and $0 \leqslant \mu_k < \infty$ may have an arbitrary dependence on the space coordinate k.

The crucial point in the derivation of the duality relations in the next section is the symmetry of H under the action of the quantum algebra $U_q[SU(2)]$,[33-35] which we already exploited in ref. 7. The generators $S^{\pm,z}$ of the quantum algebra $U_q[SU(2)]$ satisfy, by definition, the following relations:

$$[S^+, S^-] = [2S^z]_q \qquad \text{and} \qquad [S^z, S^\pm] = \pm S^\pm \tag{2.7}$$

where the expression $[x]_q$ is defined by $[x]_q = (q^x - q^{-x})/(q - q^{-1})$.

A representation in terms of Pauli matrices is given by

$$S^+ = \sum_{k=1}^{L} s_k^+(q), \qquad S^- = \sum_{k=1}^{L} s_k^-(q), \qquad S^z = \sum_{k=1}^{L} (-n_k + \tfrac{1}{2}) \tag{2.8}$$

with

$$s_k^-(q) = q^{\sum_{j=1}^{k-1}(n_j - 1)} s_k^- q^{-\sum_{j=k+1}^{L}(n_j - 1)} \tag{2.9}$$

$$s_k^+(q) = q^{\sum_{j=1}^{k-1} n_j} s_k^+ q^{-\sum_{j=k+1}^{L} n_j} \tag{2.10}$$

and $n_k = s_k^- s_k^+ = (1 - \sigma_k^z)/2$. The commutation relations can be verified using $q^{n_k} s_k^+ = s_k^+$, $s_k^+ q^{n_k} = q s_k^+$ and $q^{n_k} s_k^- = q s_k^-$, $s_k^- q^{n_k} = s_k^-$. Our representation is related to the representation given in ref. 34 by the similarity transformation

$$V = q^{-\sum_{k=1}^{L} k n_k} \tag{2.11}$$

(In ref. 33 the same representation as in ref. 34 is used, but with the replacement $q \to q^{1/2}$.)

In ref. 33 it was shown that each term $u_{k,k+1}$ in (2.3) commutes with $S^{\pm}$ and $S^z = L/2 - \sum_{k=1}^{L} n_k$. Hence

$$[H, S^{\pm}] = [H, S^z] = 0 \tag{2.12}$$

This can be derived as in ref. 33 on a purely algebraic level by expressing $u_{k,k+1}$ in terms of quantities related to the generators of the algebra, but these commutation relations are easier to check by using the explicit representation (2.6) in terms of Pauli matrices: since $[u_{k,k+1}, S^z] = [u_{k,k+1}, n_k + n_{k+1}] = 0$, one has $[u_{k,k+1}, s_j^{\pm}(q)] = 0$ for $j \neq k, k+1$. Therefore

$$[u_{k,k+1}, S^{\pm}] = [u_{k,k+1}, s_k^{\pm}(q) + s_{k+1}^{\pm}(q)]$$

and it remains to show $[u_{k,k+1}, s_k^{\pm}(q) + s_{k+1}^{\pm}(q)] = 0$. Using (2.10), one may, e.g., write

$$\begin{aligned} s_k^{+}(q) + s_{k+1}^{+}(q) &= q^{\sum_{j=1}^{k-1} n_j} (s_k^{+} q^{-n_{k+1}} + q^{n_k} s_{k+1}^{+})\, q^{-\sum_{j=k+2}^{L} n_j} \\ &\equiv q^{\sum_{j=1}^{k-1} n_j} X_{k,k+1}^{+} q^{-\sum_{j=k+2}^{L} n_j} \end{aligned}$$

where the two-site annihilation operator $X_{k,k+1}^{+}$ is given by

$$X_{k,k+1}^{+} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & q \\ 0 & 0 & 0 & 0 \end{pmatrix}_{k,k+1} \tag{2.13}$$

It is easy to check that $u_{k,k+1} X_{k,k+1}^{\pm} = X_{k,k+1}^{\pm} u_{k,k+1} = 0$, which then implies $[u_{k,k+1}, S^{\pm}] = 0$.

Note that H is also invariant under simultaneous reflection of sites $k \to L+1-k$ and particle–hole exchange and $\mu_k \to \mu_{L+1-k}$. The transpose H^T of H is related to H by the similarity transformation V, (2.11): one has

$$H^T = V^2 H V^{-2} \tag{2.14}$$

The easiest way to see this is again to use the explicit representation (2.6).

As discussed in ref. 7, one can construct from Eqs. (2.12) all the N-particle, zero-energy eigenstates $|N\rangle$ and $\langle N|$ out of the zero-particle

states $|0\rangle$ and $\langle 0|$ (in spin language these are the states with all spins up). One obtains

$$|N\rangle = \frac{1}{[N]_q!}(S^-)^N|0\rangle \quad \text{and} \quad \langle N| = \frac{1}{[N]_q!}\langle 0|(S^+)^N \tag{2.15}$$

Here $[m]_q! = [1]_q[2]_q \cdots [m]_q$. The left eigenstates assign equal weight to any N-particle configuration, as one would expect from the definition of H as generating the time evolution of a stochastic process, and are normalized so that this weight is one:

$$\langle N| = \sum_{\mathbf{n}}^{(N)} \langle \mathbf{n}|, \qquad \langle N|\mathbf{n}\rangle = 1 \qquad \forall \mathbf{n} \quad \text{with} \quad \sum_{k=1}^{L} n_k = N \tag{2.16}$$

(The upper index on the sum means that the summation runs only over states with total particle number N.) Hence, averaging over an N-particle state is performed by multiplying $\langle N|$ from the left. As noted above, multiplying with the vector $\langle s| = \sum_{\mathbf{n}} \langle \mathbf{n}| = \sum_{N=0}^{L} \langle N|$ corresponds to averaging over all states. The decomposition of the vector $\langle s|$ into pieces with fixed particle number is meaningful because of particle number conservation of the process. $\langle N|\, H = 0$ just an expression of this fact.

The normalized N-particle states

$$|N\rangle_{\text{norm}} = \mathscr{P}^{-1}|N\rangle = \mathscr{P}^{-1} q^{-N(L-1)} \sum_{\mathbf{n}}^{(N)} q^{2\sum_{k=1}^{L} k n_k} |\mathbf{n}\rangle \tag{2.17}$$

are the steady states of the inhomogeneous model, and are normalized by the factor

$$\mathscr{P} = \binom{L}{N}_q = \frac{[L]_q!}{[L-N]_q!\,[N]_q!}$$

such that they satisfy $\langle N|N\rangle_{\text{norm}} = 1$. Moreover, one can derive exact expressions for all density correlation functions in the steady state.[7]

Some Notation. (1) Instead of denoting a configuration by a set $\mathbf{n}$ of L occupation numbers $n_k = 0, 1$, one may represent a given N-particle configuration by the set C of occupied sites, $C = \{k_1, \ldots, k_N\}$, with cardinality $|C| = N$. So instead of defining the exclusion process on the state space $X = \{0, 1\}^S$, where $S = \{1, \ldots, L\}$ are the site on the lattice, one may define the process on the collection Y of subsets of S. Corresponding to a configuration $A \in Y$, we introduce the state vector $|k_1, \ldots, k_N\rangle$. In what follows we use this latter notation whenever we consider configurations with a

specific number of particles, whereas the symbol $\mathbf{n}$ is used when the number of particles is left unspecified.

(2) The transposed vector to $|k_1,...,k_N\rangle$ is denoted by $\langle k_1,...,k_N|$. Note that

$$\langle k_1,...,k_N| = \langle 0|\, s^+_{k_1} \dots s^+_{k_N} = \langle N|\, n_{k_1} \dots n_{k_N} \tag{2.18}$$

(In order to avoid confusion, we point out that $\langle N|$ is *not* the transpose of $|N\rangle$ except for $N=0, L$. This notation, however, will not give rise to problems, as only $|0\rangle$ will be used later.)

(3) In the quantum Hamiltonian language used in this paper expectation values of occupation numbers n_k are given by matrix elements of the number operator defined in (2.6). Expectation values of functions Z of the occupation numbers translate into matrix elements of these functions of the number operators, e.g., $Z = n_{k_1} \dots n_{k_N}$ for the N-point density correlation function. We shall denote expectation values of some quantity Z by $\langle Z\rangle_F$. Here the subscript F refers to the probability distribution over which one averages. For example, for the density correlation function one may either write $\langle n_{k_1} \dots n_{k_N}\rangle_F = \sum_{\mathbf{n}} n_{k_1} \dots n_{k_N} F(\mathbf{n})$, in which case the $n_{k_i} = 0, 1$ are numbers, or $\langle n_{k_1} \dots n_{k_N}\rangle_F = \langle s|\, n_{k_1} \dots n_{k_N}\, |F\rangle$. In the latter case the n_{k_i} are operators and $|F\rangle$ is defined as above by $|F\rangle = \sum_{\mathbf{n}} F(\mathbf{n})\, |\mathbf{n}\rangle$.

(4) Time-dependent expectation values are denoted $\langle Z\rangle^t_A$, where A refers to the initial configuration or a normalized average over initial configurations and the superscript t denotes the time elapsed since some initial time t_0. In such an expression the configuration $\mathbf{n}$ has an initial weight $A(\mathbf{n})$. When A_N is used rather than just A, this indicates that we consider N-particle initial configurations where the initial probability of finding the N particles on some set of lattice sites $\{k_1,...,k_N\}$ is $A_N(k_1,...,k_N)$. In the operator language introduced above such correlation functions are given by

$$\langle Z\rangle^t_{A_N} = \langle N|\, Z e^{-Ht}\, |A_N\rangle \tag{2.19}$$

where

$$|A_N\rangle = \sum_{1 \leqslant k_1 < \dots < k_N \leqslant L} A(k_1,...,k_N)\, |k_1,...,k_N\rangle \tag{2.20}$$

with $\langle N|A_N\rangle = 1$. If, e.g., initially N particles are located at sites $\{k_1,...,k_N\}$ with probability 1, then $|A_N\rangle = |k_1,...,k_N\rangle$ and $\langle Z\rangle^t_{A_N}$ is what is usually referred to as the conditional average. In this case we also use A_N to denote the set of initially occupied sites.

(5) Two-time correlation functions $\langle Z_1(t_1)\, Z_2(t_2)\rangle_A$ are defined by

$$\langle Z_1(t_1)\, Z_2(t_2)\rangle_A = \langle Z_1 e^{-H(t_1-t_2)}\, Z_2 e^{-Ht_2}\rangle_A \tag{2.21}$$

The n-time correlation functions are defined correspondingly.

(6) The indices j, k, and l always refer to (integer) space labels of the chain.

3. DUALITY RELATIONS ($q \neq 0, 1, \infty$)

In this section we establish the duality relations for the asymmetric exclusion process. Let us first remind the reader of the contents of these relations in the symmetric case and how they can be derived.

Generally, duality between two Markov processes η_t and ζ_t with state space X and Y, respectively, means

$$E^{\eta} F(\eta_t, \zeta) = E^{\zeta} F(\eta, \zeta_t) \tag{3.1}$$

for a bounded, measurable function F on $X \times Y$.[1] Thus, it is a relation between expectation values which is defined with respect to F. For the symmetric exclusion one chooses X and Y as defined above and defines for $\mathbf{n} \in X$ and $C \in Y$ the function

$$F(\mathbf{n}, C) = \begin{cases} 1 & \text{if} \quad n_k = 1 \quad \forall k \in C \\ 0 & \text{else} \end{cases} \tag{3.2}$$

For $C = \{k_1, \ldots, k_m\}$ one can write $F(\mathbf{n}, C) = n_{k_1} \ldots n_{k_m}$.

In order to show what duality for the symmetric exclusion process means, let us now assume that initially N particles are loated on a set of sites $A_N = \{l_1, \ldots, l_N\}$ and we want to compute the probability $\langle n_{k_1} \ldots n_{k_m}\rangle'_{A_N}$ of finding (any) m particles on sites $B_m = \{k_1, \ldots, k_m\}$, at time t. In words, the duality relations (3.1) with the function (3.2) states that the probability that $\mathbf{n} = 1$ at time t on all sites B_m with initial condition A_N is equal to the probability that (the original) configuration has $\mathbf{n} = 1$ on the sites occupied by an m-particle system at time t which started at sites B_m.[2] Translated into operator language, the relation reads[15]

$$\langle n_{k_1} \ldots n_{k_m}\rangle'_{A_N} = \sum_{B'_m \subset A_N} \langle n_{k'_1} \ldots n_{k'_m}\rangle'_{B_m} \tag{3.3}$$

In this expression the sum runs over all sets $B'_m = \{k'_1, \ldots, k'_m\}$ which are contained in the set A_N, i.e., the m-point correlation function $\langle n_{k_1} \ldots n_{k_m}\rangle'_{A_N}$

of the N-particle system is given by sums of m-particle correlation functions (we assume $m \geqslant N$) which are the conditional probabilities

$$\langle n_{k'_1} \dots n_{k'_m} \rangle'_{B_m} = \langle k'_1, \dots, k'_m | \, e^{-Ht} \, | k_1, \dots, k_m \rangle$$

of finding the m particles on the set of sites B'_m at time t if at time $t = 0$ they had been on sites B_m. Thus the problem of calculating an m-point correlation function of an interacting particle system of N particles has been reduced to the calculation of a correlation function of an m-particle system.

The duality relations may be written and derived as follows[15]:

$$\begin{aligned}
&\langle n_{k_1} \dots n_{k_m} \rangle'_{A_N} \\
&= \langle N | \, n_{k_1} \dots n_{k_m} e^{-Ht} \, | A_N \rangle \\
&= \langle k_1, \dots, k_m | \frac{(S^+)^{N-m}}{(N-m)!} e^{-Ht} \, | A_N \rangle \\
&= \sum_{1 \leqslant k'_1 < \dots < k'_m \leqslant L} \langle N | \, n_{k'_1} \dots n_{k'_m} \, | A_N \rangle \langle k_1, \dots, k_m | \, e^{-Ht} \, | k'_1, \dots, k'_m \rangle \\
&= \sum_{1 \leqslant k'_1 < \dots < k'_m \leqslant L} \langle N | \, n_{k'_1} \dots n_{k'_m} \, | A_N \rangle \langle k'_1, \dots, k'_m | \, e^{-Ht} \, | k_1, \dots, k_m \rangle \qquad (3.4)
\end{aligned}$$

where in the last step we used that H is a symmetric matrix. Here A_N denotes more generally a normalized superposition of initial configurations, i.e., an arbitrary initial probability distribution. For an initial configuration $\{k_1, \dots, k_m\}$ one recovers (3.3). This derivation of these relations is elementary and requires nothing but some basic concepts of the representation theory of $SU(2)$ [which is an obvious symmetry of the Hamiltonian (2.2) for $q = 1$], and the insertion of a unit operator expanded in eigenstates of the particle projectors

$$1 = \sum_{\mathbf{n}} |\mathbf{n}\rangle\langle\mathbf{n}| = \sum_{m=0}^{L} \left(\sum_{1 \leqslant k_1 < \dots < k_m \leqslant L} |k_1, \dots, k_m\rangle\langle k_1, \dots, k_m| \right) \qquad (3.5)$$

The duality relations may be understood as a consequence of the fact that the projector n_k is a spin-1/2 operator and of the corresponding selection rules of $SU(2)$: The state $\langle N|$ in the chain with L sites has total angular momentum $S = L/2$ and z component $S_z = L/2 - N$. Therefore $\langle N| \, n_{k_1} \dots n_{k_m}$ may be decomposed into states with $L/2 \geqslant S' \geqslant L/2 - m$ and $S'_z = L/2 - N$, which are obtained from m-particle states using the lowering operator S^-. Thus only m-particle amplitudes enter into the r.h.s. of Eqs. (3.3), (3.4) (see also ref. 42).

In the $U_q[SU(2)]$-symmetric case of driven diffusion n_k is not a spin-1/2 operator. Instead one has to consider another complete set of observables built by products of the operators

$$Q_k = q^{-2N_k} \tag{3.6}$$

or their discrete lattice derivatives

$$\tilde{Q}_k = (Q_k - Q_{k-1})/(q^{-2} - 1) = q^{-2N_{k-1}} n_k \tag{3.7}$$

where $N_k = \sum_{j=1}^{k} n_j$. Using the relations given in the preceding section, one can verify that they satisfy the commutation relations

$$\left[\frac{(S^+)^N}{[N]_q!}, Q_k \right] = q^{-N+1}(q^{-2} - 1)\, Q_k S_k^+ \frac{(S^+)^{N-1}}{[N-1]_q!} \tag{3.8}$$

where $S_k^+ = \sum_{j=1}^{k} s_j^+(q)$. Then, with (2.15) and (3.8), one finds

$$\langle N|\, \tilde{Q}_k = \langle 0| \left[\frac{(S^+)^N}{[N]_q!}, \tilde{Q}_k \right] = q^{-N+1} \langle k| \frac{(S^+)^{N-1}}{[N-1]_q!} \tag{3.9}$$

and, by repeated application of (3.9),

$$\langle N|\, \tilde{Q}_{k_1} \cdots \tilde{Q}_{k_m} = q^{-m(N-1)} \langle k_1, \ldots, k_m| \frac{(S^+)^{N-m}}{[N-m]_q!} \tag{3.10}$$

Here we have assumed that the $k_i \in B_m$ are pairwise different. (As before, $B_m = \{k_1, \ldots, k_m\}$.)

Now we are in a position to derive the duality relations. Multiplying (3.10) by $\exp(-Ht)\, |A_N\rangle$, using the $U_q[SU(2)]$ symmetry (2.12) of the time evolution operator, and inserting a unit operator (3.5) gives the relations

$$\begin{aligned}
\langle \tilde{Q}_{k_1} \cdots \tilde{Q}_{k_m} \rangle'_{A_N} &= \langle N|\, \tilde{Q}_{k_1} \cdots \tilde{Q}_{k_m} e^{-Ht} |A_N\rangle \\
&= \sum_{\mathbf{n}} \langle k_1, \ldots, k_m|\, e^{-Ht} |\mathbf{n}\rangle \langle \mathbf{n}|\, q^{-m(N-1)} \frac{(S^+)^{N-m}}{[N-m]_q!} |A_N\rangle \\
&= \sum_{1 \leqslant k_1' < \ldots < k_m' \leqslant L} \langle N|\, \tilde{Q}_{k_1'} \cdots \tilde{Q}_{k_m'} |A_N\rangle \\
&\quad \times \langle k_1, \ldots, k_m|\, e^{-Ht} |k_1', \ldots, k_m'\rangle \\
&= \sum_{1 \leqslant k_1' < \ldots < k_m' \leqslant L} q^{2\sum_{i=1}^{m}(k_i - k_i')} \langle N|\, \tilde{Q}_{k_1'} \cdots \tilde{Q}_{k_m'} |A_N\rangle \\
&\quad \times \langle k_1', \ldots, k_m'|\, e^{-Ht} |k_1, \ldots, k_m\rangle
\end{aligned} \tag{3.11}$$

In the last step we used the similarity transformation (2.14), giving the factor

$$q^{2\sum_{i=1}^{m}(k_i-k_i')}$$

inside the sum. Relations (3.11) are the main result of this paper. In terms of the general definition (3.1) of duality both η_t and ζ_t are the same asymmetric exclusion process defined on X and Y, respectively, and the function F, (3.2), is given by

$$F(\mathbf{n}, B_m)=\begin{cases}\prod_{i=1}^{m} q^{-2N_{k_i-1}-2k_i} & \text{if} \quad n_k=1 \quad \forall k\in B_m \\ 0 & \text{otherwise}\end{cases} \tag{3.12}$$

where $m=|B_m|$ is the cardinality of B_m and N_{k_i-1} is the number of particles up to side k_i-1 in the configuration $\mathbf{n}$. For $B_m=\{k_1,\ldots,k_m\}$ one may write

$$F(\mathbf{n}, B_m)=\prod_{i=1}^{m} q^{-2N_{k_i-1}-2k_i}n_{k_i}$$

Note that we have assumed that all $k_i\in B_m$ are pairwise different. Relations involving correlators of $\tilde{Q}_k$ where some of the k_i are identical, i.e., involving integer powers $\tilde{Q}_k^n$ of $\tilde{Q}_k$, can be obtained in the same way. Averages of Q_k^n are important for the calculation of the density dynamics. Also, in the polymer picture and the random-rod model for interface fluctuations where Q_k plays the role of a partition function, one is interested in the quenched free energy $\langle \ln Q_k\rangle$, which requires the calculation of averages values of powers of Q_k.[17,38] In the limit $q\to 1$ one recovers (3.4).

Using the commutation relations (3.8), one may also derive relations for correlation functions involving different imes. The simplest example is

$$\langle \tilde{Q}_k(t_1)\, Z(t_2)\rangle_{AX}=\sum_{k'=1}^{L}\langle k|\, e^{-H\tau}\,|k'\rangle\langle \tilde{Q}_{k'}(t_2)\, Z(t_2)\rangle_{AX} \tag{3.13}$$

with $\tau=t_1-t_2\geqslant 0$ and arbitrary Z. The equal-time correlator $\langle \tilde{Q}_{k'}(t_2)\, Z(t_2)\rangle_{AX}$ on the r.h.s. of (3.13) may be simplified using relations (3.11).

It is interesting to note that the operator $\mathbf{Q}_k=q^k Q_k$ is the lattice analogue of the Hopf–Cole transformation on the KPZ equation[20]

$$\frac{\partial}{\partial t}h(x,t)=\frac{\partial^2}{\partial x^2}h(x,t)+\nu\left(\frac{\partial}{\partial x}h(x,t)\right)^2+\eta(x,t) \tag{3.14}$$

for the continuum height variable $h(x, t)$ for surface growth in $1+1$ dimensions. In (3.14) $\eta(x, t)$ is the usual Gaussian white noise. [In what follows, we distinguish functions $f(x)$ of the real number x in the continuum from lattice functions f_k, where k is an integer number, by writing x as an argument of the function rather than as a subscript.] The (inverse) Hopf–Cole transformation[36] $h(x, t) \to \mathbf{Q}(x, t) = \exp[\nu h(x, t)]$ relates (3.14) to a linear diffusion equation

$$\frac{\partial}{\partial t}\mathbf{Q}(x, t) = \frac{\partial^2}{\partial x^2}\mathbf{Q}(x, t) + \eta(x, t)\,\mathbf{Q}(x, t) \tag{3.15}$$

The correspondence to the lattice problem discussed above can be understood as follows. Consider a surface over a one-dimensional discrete space where height differences between neighboring lattice sites may differ only by units of ± 1 (Fig. 1). In the mapping to the particle model a configuration of particles on the chain represents the shape of the surface at some time t by saying that a particle at site k ($k = 0, 1, 2, \ldots, L$) represents a height difference $h_k - h_{k-1} = -1$ between sites k and $k-1$, while a vacancy represents a local height difference $+1$. The stochastic dynamics of the model is given by the Hamiltonian (2.2). Hopping of a particle to the right corresponds to growth of the surface height at a local minimum, while hopping to the left corresponds to shrinking of the height at a local maximum (see Fig. 1). The average height $\langle h_k(t)\rangle$ of the surface at site k is therefore given by the expectation value $2\langle S_k^z\rangle$ with $2S_k^z = k - 2N_k$. If the system is

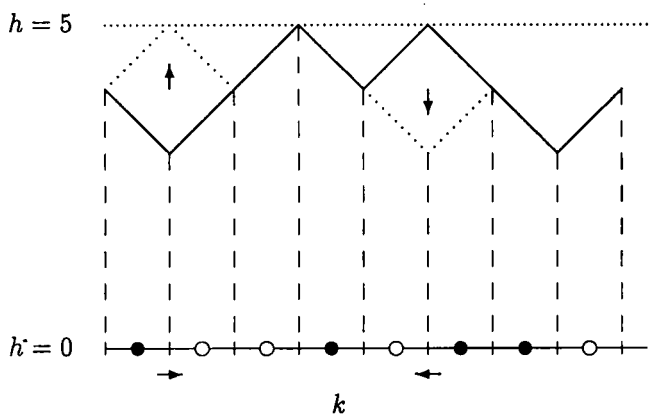

Fig. 1. The mapping between the restricted interface and the particle exclusion process: we show a possible interface configuration and the corresponding particle occupancies on a lattice with sites labeled by k. The indicated flips in the interface corresponds to particles hopping on the lattice, marked by horizontal arrows.

homogeneous, i.e., $\mu_j = 1/2\ \forall j$, the average height satisfies the differential-difference equation

$$\frac{\partial}{\partial t}\langle h_k(t)\rangle = \frac{q+q^{-1}}{2}\left[\langle h_{k+1}(t)\rangle + \langle h_{k-1}(t)\rangle - 2\langle h_k(t)\rangle\right] + \frac{q-q^{-1}}{2}\left\{1 - \langle [h_{k+1}(t)-h_k(t)][h_k(t)-h_{k-1}(t)]\rangle\right\} \quad (3.16)$$

which is a discretized, averaged KPZ equation. On the other hand, the expectation value of the lattice operator

$$\mathbf{Q}_k = q^k Q_k = q^{S_k^z} = \exp(f h_k)$$

satisfies

$$\frac{\partial}{\partial t}\langle Q_k(t)\rangle = \langle \mathbf{Q}_{k+1}(t)\rangle + \langle \mathbf{Q}_{k-1}(t)\rangle - (q+q^{-1})\langle \mathbf{Q}_k(t)\rangle \quad (3.17)$$

This is indeed a discretized form of (3.15) with $\langle \eta \mathbf{Q}\rangle = (q+q^{-1}-2)\langle \mathbf{Q}\rangle$. Therefore the transformation $2S_k^z \to q^{2S_k^z}$ (or $-2N_k \to Q_k$) may indeed be considered an (inverse) lattice Hopf–Cole transformation.

Note that the one-point functions $\langle \mathbf{Q}_k\rangle$ and $\langle Q_k\rangle$ on the lattice are given in terms of one-particle excitations. As pointed out in ref. 7, in the infinite-volume limit the one-particle sector is characterized by a dynamical exponent $z=2$. We conclude that the *bulk* dynamics of these exponentials has $z=2$, as opposed to the dynamics of the density variables themselves, which has $z=3/2$.[4,24] We have emphasized here that this distinction is valid for bulk quantities. In the next section we shall see that the situation near the boundary of the system needs special attention. Note that the dynamical exponent $z=2$ for quantities related to $\mathbf{Q}(x)$ may also be derived for the continuum theory.[37]

4. THE TOTALLY ASYMMETRIC CASE

Here we study the totally asymmetric model where particles hop in only one direction. We choose $q=0$, corresponding to hopping to the right, i.e., particles may move with some rate μ_k from site k to site $k+1$, but not backward. Renormalizing the time scale by a factor $q+q^{-1}$, we find that the Hamiltonian (2.2) becomes

$$H = -\sum_{k=1}^{L-1} \mu_k (s_k^+ s_{k+1}^- - n_k(1-n_{k+1})) \quad (4.1)$$

and commutes with the operators $T^{\pm}$, T^0, and $\bar{T}^0$ defined by

$$\begin{aligned} T^+ &= \sum_{k=1}^{L} \left(\prod_{j=1}^{k-1} (1-n_j)\, s_k^+ \right) \\ T^- &= \sum_{k=1}^{L} \left(s_k^- \prod_{j=k+1}^{L} n_j \right) \\ T^0 &= \prod_{k=1}^{L} (1-n_k) \\ \bar{T}^0 &= \prod_{k=1}^{L} n_k \end{aligned} \tag{4.2}$$

They satisfy the relations

$$\begin{aligned} [T^+, T^-] &= T^0 - \bar{T}^0 \\ [T^0, \bar{T}^0] &= 0 \\ T^+ T^0 &= \bar{T}^0 T^+ = 0 \\ T^0 T^- &= T^- \bar{T}^0 = 0 \end{aligned} \tag{4.3}$$

Note that H given by (4.2) also commutes with S^z, (2.8), expressing particle number conservation.

The (normalized) N-particle steady states $|N\rangle$ of H, (4.1), are given by

$$|N\rangle = (T^-)^N |0\rangle = |L+1-N, L+2-N, \ldots, L\rangle \tag{4.4}$$

These are simply the states with N particles occupying the N rightmost sites of the chain. On the other hand, averaging is done as for finite asymmetry q by multiplying with the left N-particle, zero-energy eigenstate

$$\langle N| = \langle 0| (T^+)^N = \sum_{1 \leqslant k_1 < \ldots < k_N \leqslant L} \langle k_1, \ldots, k_N| \tag{4.5}$$

The duality relations for $q=0$ can be obtained by taking the limit in (3.11). For practical purposes, however, this is tedious. Instead we rederive the relation for the one-point function using the algebra (4.2). We study expectation values of the operators

$$Y_k = \prod_{j=k}^{L} (1-n_j) \tag{4.6}$$

giving the probability of finding the last $L-k+1$ sites in the chain unoccupied. Furthermore, we define

$$\tilde{Y}_k = Y_{k+1} - Y_k = n_k \prod_{j=k+1}^{L} (1-n_j) \qquad (1 \leqslant k \leqslant L-1) \tag{4.7}$$

A short calculation shows that

$$(T^+)^N Y_k = Y_k T_{k-1}^+ (T^+)^{N-1} \tag{4.8}$$

where $T_k^+ = \sum_{l=1}^{k} [\prod_{j=1}^{l-1} (1-n_j)\, s_l^+]$ and therefore

$$\langle N|\, \tilde{Y}_k = \langle k|\, (T^+)^{N-1} \tag{4.9}$$

This leads to the relation

$$\begin{aligned} \langle \tilde{Y}_k \rangle_{A_N} &= \sum_{k'=1}^{L} \langle k|\, e^{-Ht}\, |k'\rangle \langle k'|\, (T^+)^{N-1}\, |A_N\rangle \\ &= \sum_{k'=1}^{L} \langle k|\, e^{-Ht}\, |k'\rangle \langle N|\, \hat{Y}_{k'}\, |A_N\rangle \end{aligned} \tag{4.10}$$

This relation expresses the fact that the probability of finding a particle followed by holes only is independent of the motion of all the other particles to the left of the rightmost particle. This result may be derived directly from the master equation (or other reasonable definitions of the process), but we find it interesting that this statement follows from the quantum algebra symmetry of the process and that it is the analogue of the less obvious duality relations for finite asymmetry and of the usual duality relations for the symmetric case. Note that for $N \neq 0$ one has

$$\langle Y_k \rangle_{A_N} = \sum_{j=1}^{k-1} \langle \tilde{Y}_j \rangle_{A_N} \tag{4.11}$$

since $\langle Y_1 \rangle_{A_N} = 0$ except in the trivial zero-particle sector.

Using the symmetry under parity and particle–hole exchange, one finds similar relations for

$$B_k = \prod_{j=1}^{k} n_j \tag{4.12}$$

and

$$\tilde{B}_k = B_{k-1} - B_k = (1-n_k) \prod_{j=1}^{k-1} n_j \qquad (2 \leqslant k \leqslant L) \tag{4.13}$$

The corresponding relations read

$$\langle \tilde{B}_k \rangle_{A_N} = \sum_{k'=1}^{L} \langle L+1-k| \, e^{-Ht} \, |L+1-k'\rangle \langle N| \, \tilde{B}_{k'} \, |A_N\rangle \tag{4.14}$$

Note that except in the trivial case of a completely filled system ($N=L$) one has

$$\langle B_k \rangle_{A_N} = \sum_{j=k+1}^{L} \langle \tilde{B}_j \rangle_{A_N} \tag{4.15}$$

Other relations of this kind may be obtained for the operators $\tilde{Y}_{k_1 \ldots k_m}$, which are strings of projectors on holes and particles ranging fromsite k_1 to site L, e.g.,

$$\tilde{Y}_{k_1, k_2} = n_{k_1}(1-n_{k_1+1}) \ldots (1-n_{k_2-1}) \, n_{k_2}(1-n_{k_2+1}) \ldots (1-n_L)$$

One can show that expectation values for these operators are given in terms of m-point correlation functions. This simply expresses the fact that the motion of the m rightmost particles is independent of the motion of the remaining $N-m$ particles to their left.

As an application we compute the probability $p_l(k, t) = \langle \tilde{B}_k \rangle_{A_N(l)}$ of finding the first $k-1$ sites of a chain occupied, followed by a vacancy. As initial state we take an arbitrary N-particle state where the first $l-1$ sites are occupied, followed by a vacancy on site l. For any such state one has $p_l(k, t=0) = \langle N| \, \tilde{B}_k \, |A_N(l)\rangle = \delta_{k,l}$ and therefore

$$p_l(k, t) = \langle L+1-k| \, e^{-Ht} \, |L+1-l\rangle \tag{4.16}$$

The matrix element on the r.h.s. of (4.16) cannot be calculated by inserting a complete set of eigenstates of H, since H is not diagonizable in the totally asymmetric limit. However, it is easy to compute the action of H on the one-particle state $|L+1-l\rangle$ directly. One finds ($2 \leqslant k \leqslant L$)

$$\langle L+1-k| \, H^n \, |L+1-l\rangle = \begin{cases} (-1)^{k-l} \binom{n}{l-k}, & l \geqslant k \\ 0, & l < k \end{cases} \tag{4.17}$$

Expanding the exponential in (4.16) in powers of t yields

$$p_l(k, t) = \Theta_r \frac{t^r}{r!} e^{-t} \tag{4.18}$$

where we have defined $r = l - k \geqslant 0$ and the Heaviside step function $\Theta_r = 1$ for $r \geqslant 0$ and $\Theta_r = 0$ for $r < 0$. This result depends on the length L of the system and the number o particles $N < L$ only insofar as we assume that $2 \leqslant k \leqslant L$ and $2 \leqslant l \leqslant N + 1$.

The quantity $\langle j(1, t) \rangle_{A_N(l)} = -p_l(2, t)$ is the time derivative of the density $\rho(1, t) = \langle n_1 \rangle_{A_N(l)}$ and therefore the current at the left boundary of the system. Its absolute value has a maximum (as a function of t) for $t = r = l - 2$. In order to study the scaling behavior of the current we set $l - 2 = t - R$ nd take the limit $t, l \to \infty$, keeping the ratio $u = R^2/t$ fixed. In this scaling limit we find

$$\langle j(1, t) \rangle_{A_N(l)} \sim -\frac{1}{\sqrt{2\pi t}} e^{-R^2/t} \tag{4.19}$$

corresponding to a dynamical exponent $z = 2$ and *not* $z = 3/2$ as for bulk density correlations.[4.9,24]

This result may be interpreted in terms of height variables as follows: We define as in Section 3 a one-dimensional lattice with site labels k running from 0 to L and with height variables h_k and height gradients $h_k - h_{k-1} = 1 - 2n_k = \pm 1$. We set $h_0 = 0$ and take an initial height profile which first runs $l - 1$ steps downward and followed by a step upward at site l. We do not specify the initial height configuration for sites $l' > l$ (Fig. 2). So we have at time $t = 0$

$$h_k = \begin{cases} -k, & 0 \leqslant k \leqslant l - 1 \\ -j + 2, & k = l \\ \text{arbitrary} & k \geqslant l + 1 \end{cases} \tag{4.20}$$

where "arbitrary" is to be understood within the constraint on the gradient. The boundary conditions for the particle model are such that no particles can enter or leave the system, corresponding to keeping $h_0 = 0$ fixed for all times. Thus $\langle h_k(t) \rangle = k - 2 \sum_{j=1}^{k} \langle \eta_j \rangle$ and in particular

$$\frac{\partial}{\partial t} \langle h(1, t) \rangle_{A_N(l)} = -2 \langle j(1, t) \rangle_{A_N(l)} \tag{4.21}$$

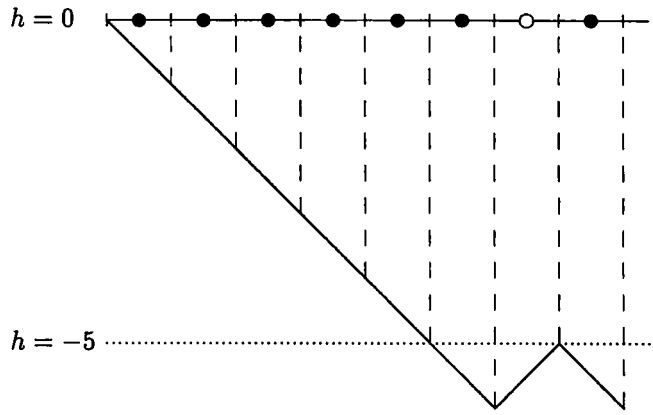

Fig. 2. Initial interface and corresponding particle configuration at the left boundary at time $t=0$ with $l=6$ as discussed in the text. The interface has a negative slope $h_k - h_{k-1} = -1$ up to site 6, followed by a positive bump at site 7.

Thus $\langle j(1,t)\rangle_{A_N(l)}$ may be interpreted as a special boundary height correlation function with dynamical exponent $z=2$.

5. OTHER ASYMMETRIC EXCLUSION PROCESSES

The main ingredient in our derivation in Section 3 was the fact the time evolution operator, or, rather the matrices u_j, (2.3), commute with the generators $S^{\pm,z}$, (2.8), of the quantum algebra. Therefore, the duality relations (3.11) hold for *any* stochastic process which can be defined in terms of linear combinations of the $u_{k,k+1}$ and their products. In order to illustrate this, we define two other asymmetric exclusion processes for which the duality relations hold.

5.1. Asymmetric Exclusion Process with Next-Nearest Neighbor Interaction

We consider a stochastic process in one dimension where particles may hop one or two sites according to the following rules. To each initial configuration of any three neighboring sites $(k, k+1, k+2)$ (where $1 \leqslant k \leqslant L-2$) we give the configurations into which the system evolves after an infinitesimal time step. The coefficients given with the final states are the relative probabilities of these events:

$$
\begin{aligned}
&(100) \to q^3(010) + q(001) \\
&(010) \to q(100) + q^3(001) \\
&(001) \to q^{-1}(100) + q(010) \\
&(110) \to q(101) + q^3(011) \\
&(101) \to q(110) + q^3(011) \\
&(011) \to q^{-3}(110) + (q + q^{-1} - q^{-3})(101)
\end{aligned}
\tag{5.1}
$$

The first line means that a particle followed by two vacancies hops with rate q^3 one lattice unit to the right and with rate q by two lattice units. In order to ensure positivity of all rates we assume $q + q^{-1} - q^{-3} \geqslant 0$. The configurations (000) and (111) do not change. The Hamilton operator defining the stochastic time evolution is given by

$$
\tilde{H} = \sum_{k=1}^{L-2} v_{k,k+1,k+2} \tag{5.2}
$$

where

$$
\begin{aligned}
v_{k,k+1,k+2} = &(1 + q^2)(u_{k+1,k+2}) \\
&- q u_{k,k+1} u_{k+1,k+2} - q^{-1} u_{k+1,k+2} u_{k,k+1}
\end{aligned}
\tag{5.3}
$$

Obviously $\tilde{H}$ commutes with the generators (2.8) of the quantum algebra $U_q[SU(2)]$.

By choosing other linear combinations

$$
\alpha u_{k,k+1} + \beta u_{k+1,k+2} + \gamma u_{k,k+1} u_{k+1,k+2} + \delta u_{k+1,k+2} u_{k,k+1}
$$

one can define other processes involving next-nearest-neighbor interactions. Note that since the $u_{k,k+1}$ are generators of a Temperley–Lieb algebra with relations[3]

$$
u_{k,k+1}^2 = (q + q^{-1})\, u_{k,k+1}, \qquad u_{k,k+1} u_{k\pm1,k+1\pm1} u_{k,k+1} = u_{k,k+1}
$$

there are no other independent products of $u_{k,k+1}$ and $u_{k+1,k+2}$.

5.2. Asymmetric Exclusion Process with Parallel Updating

So far we have considered stochastic processes defined in continuous time. One may also define the asymmetric exclusion process in discrete time

by choosing a parallel updating scheme as follows: We divide each (discrete) time step into two half time steps. In the first half time step the chain of L sites is divided into pairs (1, 2), (3, 4), etc., of sites. If both sites in a pair are occupied or empty, they remain so. If the left site $2k-1$ in a pair is occupied and site $2k$ empty, then the particle hops with probability $\mu_k q$ and remains on site $2k-1$ with probability $1-\mu_k q$. On the other hand, if the left site $2k$ in the pair is occupied and site $2k-1$ empty, the particle hops with probability $\mu_k q^{-1}$. These rules are applied in parallel to all pairs. In the second half time step the pairing of the lattice is shifted by one unit such that the pairs are the sites (2, 3), (4, 5), etc., and the same rules are applied again. Here the quantities $\mu_k q$ and $\mu_k q^{-1}$ are probabilities and therefore have to satisfy $0 \leqslant \mu_k q,\ \mu_k q^{-1} \leqslant 1$.

The discrete time evolution of this system may be expressed in terms of a transfer matrix

$$T = T^{\text{odd}} T^{\text{even}} = \prod_{k=1}^{L/2} T_{2k-1} \prod_{k=1}^{L/2-1} T_{2k} \tag{5.4}$$

with the nearest-neighbor hopping matrices

$$T_k = 1 - \mu_k u_{k,k+1} \tag{5.5}$$

where $u_{k,k+1}$ is the same matrix as in (2.3). This is the transfer matrix of an inhomogeneous asymmetric six-vertex model.[10,15] Again, obviously T commutes with the generators $S^{\pm,z}$, (2.8), of the quantum algebra $U_q[SU(2)]$ and the duality relations (3.11) and (3.13) are valid with $\exp(-Ht)$ replaced by T^t.

6. CONCLUSIONS

Using the $U_q[SU(2)]$ symmetry of the asymmetric exclusion process with reflective boundary conditions, we derived relations (3.11) expressing m-point correlation functions of the operators $\tilde{Q}_k$, (3.7), in the N-particle sector in terms of simpler correlation functions for a system of only m particles. These relations are the generalization of the duality relations for the symmetric exclusion process. This symmetry may also be used for the derivation of relations involving correlation functions for different times, an example of which is given in (3.13). In the mapping to the growth model discussed in Section 3 the closely related operator $\mathbf{Q}_k$ is the lattice analogue of the Hopf–Cole transformation on the KPZ equation and the dynamics of this operator is, as in the continuum case, characterized by a dynamical exponent $z = 2$.

In the totally asymmetric case we have derived explicit expression for the boundary density correlation functions $\langle n_1 n_2 \dots n_k \rangle$ and $\langle (1-n_k)(1-n_{k+1}) \dots (1-n_L) \rangle$. We have found a dynamical exponent $z=2$ in the expression (4.19) for the boundary current. On the other hand, one expects for density correlations in the bulk $z=3/2$. This observation raises two questions: the first is whether the dynamical exponent for boundary correlations is $z=2$ in general, and second, whether one would find this exponent also in the continuum growth model with fixed boundary height.

Finally, we have pointed out that the duality relations (3.11) and (3.13) hold for any $U_q[SU(2)]$-symmetric, single-species exclusion process. The essential point, however, is not that we have considered only exclusion processes involving one type of particle, but the $U_q[SU(2)]$ symmetry of the time evolution. Therefore, other duality relations may be derived for $U_q[SU(2)]$-symmetric exclusion processes involving more than one species of particles. Possible candidates for two-species processes may be among the spin-1 Hamiltonians studied in refs. 8, 44, and 45.

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